

NUMERICAL CALCULATIONS OF THE WIGNER ROTATION

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ABSTRACT

We have introduced the Wigner rotation. We represented special and most general Lorentz transformations. The velocity addition formula for special and most general Lorentz transformations are clearly explained. We have derived the formula of Wigner rotation in two different ways. We have calculated the numerical values of Wigner rotation for different cases. The graph of the Wigner rotation with respect to different velocities has plotted

KEYWORDS: Special Lorentz Transformation; Most General Lorentz Transformation; Lorentz Transformation; Velocity Addition Formula Most General Lorentz Transformation; Velocity Addition Formula Special Lorentz Transformation; Wigner Rotation. PACS: 03.30. + P

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1. INTRODUCTION

Wigner rotation results from the formation of two non-collinear Lorentz boosts that is not a pure boost, but is the composition of a boost and a rotation. This rotation is known as Thomas rotation, Thomas–Wigner rotation or Wigner rotation. The rotation was discovered by Thomas in 1926[1] and derived by Wigner in 1939[2]. When a sequence of non-collinear boosts returns the spatial origins of a sequence of inertial frame to the starting point, then the sequence of Wigner rotations combine to produce a net rotation called the Thomas precession [3]. Consider four inertial frames of references S , π , μ and e where the frame S is the lab frame which is at rest and the frame π is moving with uniform velocity \vec{U} with respect to S , μ is moving with uniform velocity \vec{V} with respect to π , e is moving with uniform velocity \vec{W} respect to μ . We want to find the velocity of the electron with respect to lab frame S . There are two ways to get the velocity of electron with respect to lab frame. Firstly, we can take the Lorentz sum of \vec{U} and \vec{V} i.e. $(\vec{U} \oplus \vec{V})$ (where \oplus denotes the Lorentz sum) and then we can take the Lorentz sum of $(\vec{U} \oplus \vec{V})$ and \vec{W} i.e. $(\vec{U} \oplus \vec{V}) \oplus \vec{W}$. Secondly, we can take the Lorentz sum of \vec{V} and \vec{W} i.e. $(\vec{V} \oplus \vec{W})$ and then, we can take the Lorentz sum of \vec{U} and $(\vec{V} \oplus \vec{W})$ i.e. $\vec{U} \oplus (\vec{V} \oplus \vec{W})$. The angle between these two velocities $(\vec{U} \oplus \vec{V}) \oplus \vec{W}$ and $\vec{U} \oplus (\vec{V} \oplus \vec{W})$ is Wigner rotation.

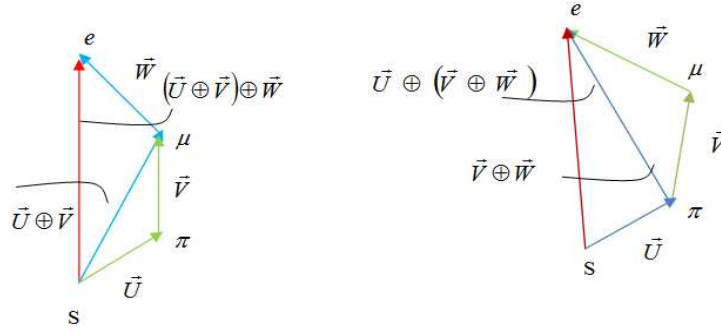


Figure 1: Wigner Rotation of π , μ and e Decay Change

We can also explain the Wigner rotation by the following ways. Let us consider Alice is moving with velocity \vec{v}_1 with respect to mission control and Bob is moving with velocity \vec{v}_2 respect to Alice, as shown in Figure 2. Unfortunately, mission control cannot directly observe the velocity of Bob. The relativistic combination of velocities is how we deduce the velocity \vec{v}_{21} of Bob, as seen by mission control, using the velocities \vec{v}_1 and \vec{v}_2 . We must be clear that \vec{v}_2 is measured in Alice's rest frame whilst \vec{v}_1 and \vec{v}_{21} are measured in mission control's rest frame. We derive a simple formula for this velocity \vec{v}_{21} , and it is quantitative results the relativistic combination of velocities \vec{v}_1 and \vec{v}_2 [4]. Mission control observes Bob as the spacecraft labeled B21, and to be moving at velocity, but pointing in a direction rotated by the Wigner rotation angle Ω .

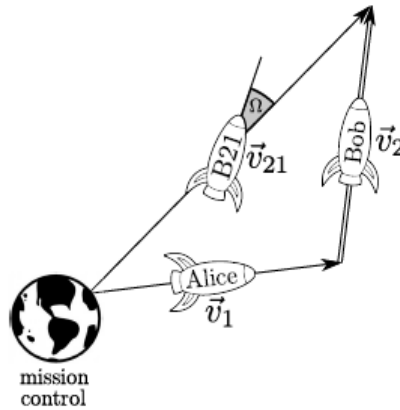


Figure 2: The Wigner Rotation of Bob with Respect to Mission Control

2. LORENTZ TRANSFORMATION

The transformation which relates the observations of position and time made by the two observers in two different inertial frames is known as Lorentz transformation.

2.1. Special Lorentz Transformation

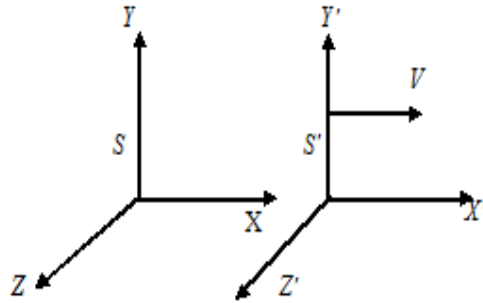


Figure 3: The Frame S is at Rest and the Frame S' is Moving with Respect to S with Uniform Velocity v Along x-Axis

Let us consider two inertial frame of references S and S', where the frame S is at rest and the frame S' is moving along x-axis with velocity v with respect to S frame. The space and time coordinates of S and S' are (x, y, z, t) and (x', y', z', t') respectively. The relation between the coordinates of S and S', which is called special Lorentz Transformation, can be written as [5-8]

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (1)$$

And

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (2)$$

Special Lorentz Transformation is one dimensional. The velocities of moving frames are along x-axis. So, there is no Wigner rotation Special Lorentz Transformation.

2.2. Most General Lorentz Transformation

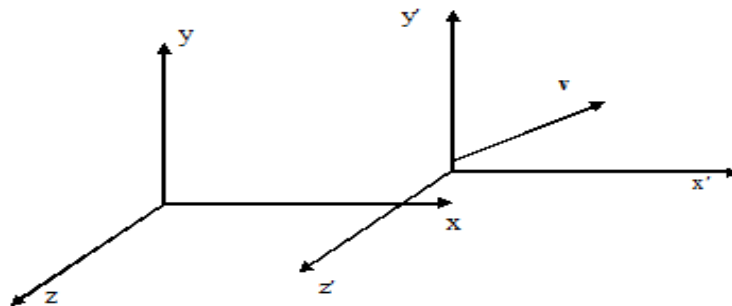


Figure 4: The Frame S is at Rest and the Frame S' is Moving with Respect to S with Uniform Velocity \vec{v} Along any Arbitrary Direction

When the velocity \vec{v} of S' with respect to the S is not along x-axis i.e. the velocity \vec{v} has three components v_x , v_y and v_z then the relation between the coordinates of S and S' , which is called most general Lorentz transformation, can be written as [5, 8-10]

$$\vec{X}' = \vec{X} + \vec{v} \left[\left\{ (\vec{X} \cdot \vec{v}) / v^2 \right\} \left\{ (1 - v^2/c^2)^{-1/2} - 1 \right\} - t(1 - v^2/c^2)^{-1/2} \right] \quad (3)$$

$$t' = (1 - v^2/c^2)^{-1/2} (t - \vec{X} \cdot \vec{v} / c^2)$$

and

$$\vec{X} = \vec{X}' + \vec{V}' \left\{ \frac{(\vec{X}' \cdot \vec{V}')}{V'^2} (\gamma - 1) - t' \gamma \right\} \quad (4)$$

$$t = \gamma(t' - \vec{X}' \cdot \vec{V}')$$

$$\text{Where, } \gamma = \left(1 - \frac{V^2}{c^2} \right)^{-\frac{1}{2}}, c = 1 \text{ and } \vec{V}' = -\vec{V}$$

Here \vec{X} and \vec{X}' is the space part of the S and S' frame respectively.

3. RELATIVISTIC VELOCITY ADDITION FORMULA OF LORENTZ TRANSFORMATION

3.1 Relativistic Velocity Addition Formula of special Lorentz Transformation

Consider three inertial frames of reference S , S' and S'' where the frame S is at rest and the frame S' is moving along X-axis with velocity v with respect to S frame and the frame S'' is moving along X-axis with velocity u with respect to S' . Then,

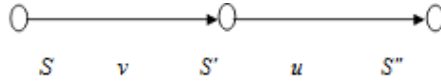


Figure 5: Velocity Addition for Special Lorentz Transformation

From Equation (2), we have

$$\frac{x}{t} = \frac{x' + v t'}{t' + \frac{v x'}{c^2}}$$

Dividing numerator and denominator by t' we get.

$$\text{or, } w = \frac{\frac{x'}{t'} + v}{1 + \frac{v}{c^2} \frac{x'}{t'}}$$

Putting $u = \frac{x'}{t'}$ we have

$$w = \frac{u + v}{1 + u v} \quad (5)$$

Equation (5) is the relativistic or Einstein velocity addition theorem for special Lorentz transformation.

3.2 Relativistic Velocity Addition Formula of Most General Lorentz Transformation

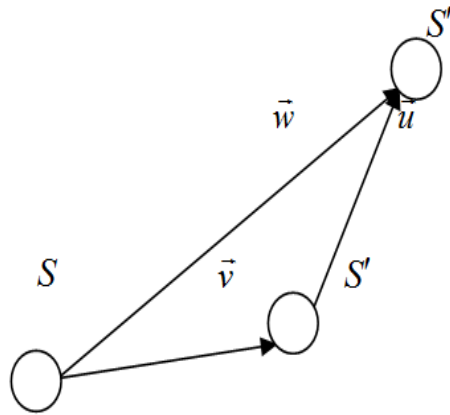


Figure 6: Velocity Addition for Most General Lorentz Transformation

Consider three inertial frames of reference S , S' and S'' where the frame S is at rest and the frame S' is moving with velocity \vec{v} with respect to S frame and the frame S'' is moving with velocity \vec{u} with respect to S' frame. From the equation (4), we have

$$\frac{\vec{X}}{t} = \frac{\vec{X}' + \vec{v} \left\{ \frac{(\vec{X}' \cdot \vec{v})}{v^2} (\gamma - 1) + t' \gamma \right\}}{\gamma(t' + \vec{X}' \cdot \vec{v})} \quad (6)$$

Dividing numerator and denominator of equation (6) by t' we get,

$$\vec{w} = \frac{\vec{u} + \vec{v} \left\{ \frac{(\vec{u} \cdot \vec{v})}{v^2} (\gamma - 1) + \gamma \right\}}{\gamma(1 + \vec{u} \cdot \vec{v})} \quad (7)$$

$$\text{Where } \vec{w} = \frac{\vec{X}}{t}, \vec{u} = \frac{\vec{X}'}{t'}, \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Equation (7) is the relativistic velocity addition theorem for most general Lorentz transformation.

3.3 Relativistic Velocity Addition Formula of Most General Lorentz Transformation in the Case of Perpendicular Velocities

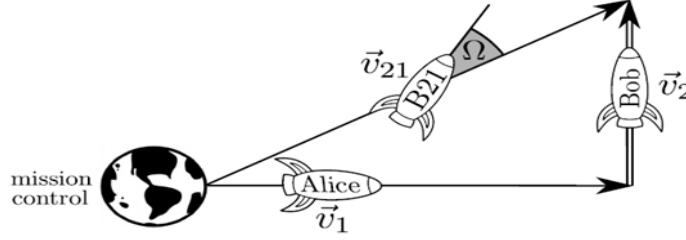


Figure 7: Perpendicular Relativistic Combination of Velocities between Bob and Alice

Let us consider the special cases of perpendicular velocities between Alice and Bob, Alice is moving with velocity \vec{v}_1 with respect to mission control and Bob is moving with velocity \vec{v}_2 with respect to Alice, as shown in Figure 7. The velocity of Bob is supposed to read \vec{v}_{21} by mission control using equation (7) it can be written as,

$$\vec{v}_{21} = \frac{\vec{v}_2 + \vec{v}_1 \left[\left\{ \frac{v_1 v_2 \cos 90^\circ}{v_2^2} \right\} \{\gamma_1 - 1\} + \gamma_1 \right]}{\gamma_1 (1 + v_1 v_2 \cos 90^\circ)} \quad \text{Where } \gamma_1 = \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}} \text{ and } c=1.$$

$$\vec{v}_{21} = \frac{\vec{v}_2 + \vec{v}_1 (\gamma_1)}{\gamma_1}$$

$$\vec{v}_{21} = \vec{v}_1 + \frac{\vec{v}_2}{\gamma_1} = \vec{v}_1 + \vec{v}_2 \sqrt{1 - v_1^2} \quad (8)$$

Similarly, using equation (7) the relativistic combination of velocities \vec{v}_{12} can be written as,

$$\vec{v}_{12} = \frac{\vec{v}_1 + \vec{v}_2 \left[\left\{ \frac{v_1 v_2 \cos 90^\circ}{v_1^2} \right\} \{\gamma_2 - 1\} + \gamma_2 \right]}{\gamma_2 (1 + v_1 v_2 \cos 90^\circ)} \quad \text{Where } \gamma_2 = \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}}$$

$$\vec{v}_{12} = \frac{\vec{v}_1 + \vec{v}_2 (\gamma_2)}{\gamma_2}$$

$$\vec{v}_{12} = \vec{v}_2 + \frac{\vec{v}_1}{\gamma_2} = \vec{v}_2 + \vec{v}_1 \sqrt{1 - v_2^2} \quad (9)$$

These formulas are extremely useful to introduce the concept of relativistic ally combining velocities.

3.4 Relativistic Velocity Addition Formula of Most General Lorentz Transformation in the Case of Parallel velocities

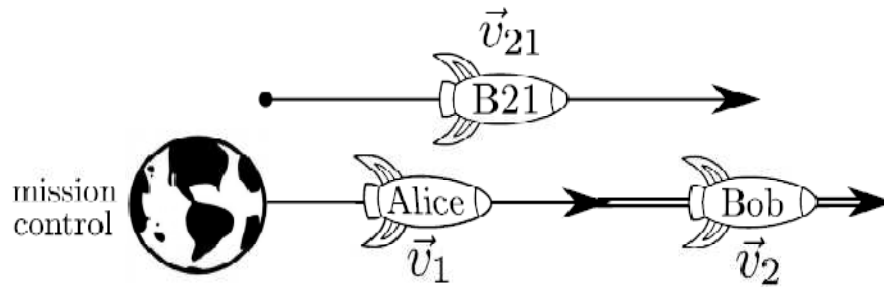


Figure 8: Parallel Relativistic Combination of Velocities between Bob and Alice

The special cases of parallel velocities between Alice velocity \vec{v}_1 and Bob velocity \vec{v}_2 are shown in Figure 8. The relativistic combination of parallel velocities formula is usually given in textbooks [1, 16-19]. Using equation (7) the velocities \vec{v}_{21} can be written as,

$$\vec{v}_{21} = \frac{\vec{v}_2 + \vec{v}_1 \left[\frac{v_1 v_2 \cos \theta^0}{v_2^2} \right] \{ \gamma_1 - 1 \} + \gamma_1}{\gamma_1 (1 + v_1 v_2 \cos \theta^0)} \text{ Where } \gamma_1 = \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}} \text{ and } c=1.$$

$$\vec{v}_{21} = \frac{\vec{v}_1 + \vec{v}_2}{1 + \vec{v}_1 \cdot \vec{v}_2}$$

Using equation (7) the relativistic combination of velocities \vec{v}_{12} can be written as,

$$\vec{v}_{12} = \frac{\vec{v}_1 + \vec{v}_2 \left[\frac{v_1 v_2 \cos \theta^0}{v_1^2} \right] \{ \gamma_2 - 1 \} + \gamma_2}{\gamma_2 (1 + v_1 v_2 \cos \theta^0)} \text{ Where } \gamma_2 = \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}}$$

$$\vec{v}_{12} = \frac{\vec{v}_1 + \vec{v}_2}{1 + \vec{v}_1 \cdot \vec{v}_2}$$

So we can write,

$$\vec{v}_{21} = \vec{v}_{12} = \frac{\vec{v}_1 + \vec{v}_2}{1 + \vec{v}_1 \cdot \vec{v}_2} \quad (10)$$

Here $c = 1$. The magnitudes of the two combined velocities \vec{v}_{21} and \vec{v}_{12} are the same, and hence there will be no resulting Wigner rotation or Thomas precession.

4. DERIVATION OF THE FORMULA OF WIGNER ROTATION

We have derived the formula of Wigner rotation in two ways. Initially we have calculated change of Wigner rotation of Bob with respect to mission control. Later, we have calculated Wigner rotation of π , μ and e .

4.1 Method One Calculation of Wigner Rotation

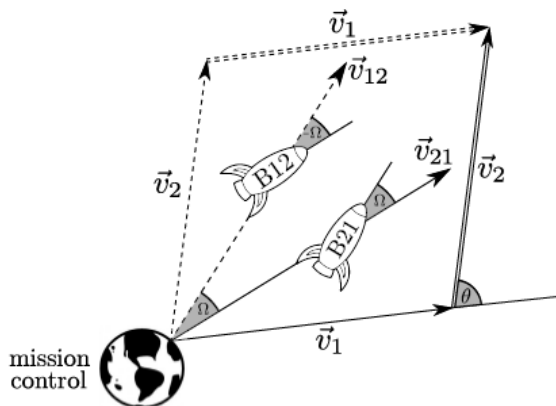


Figure 9: A More Correct Interpretation of the Relativistic Combination of Velocities

In figure 9, Alice is moving with velocity \vec{v}_1 with respect to mission control while Bob is moving with uniform velocity \vec{v}_2 with respect to Alice which is shown by the solid line. When mission control observes Bob velocity, it is supposed to read \vec{v}_{21} , but it observes a rotation of angle Ω which is the Wigner rotation angle. The dashed lines indicate Alice has velocity \vec{v}_2 as measured by mission control, and Bob has velocity \vec{v}_1 as measured by Alice as shown. When mission control observes Bob velocity it is supposed to read \vec{v}_{12} . But it also observes a rotation of angle Ω . This angle is also known as Wigner rotation.

Let us consider with the case of \vec{v}_1 and \vec{v}_2 being perpendicular, and hence with the relativistic combined velocities \vec{v}_{21} and \vec{v}_{12} as defined by (8) and (9) respectively. We can write,

$$\vec{v}_{21} \neq \vec{v}_{12}, \quad \text{But} \quad |\vec{v}_{12}| = |\vec{v}_{21}| = \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2} \quad (11)$$

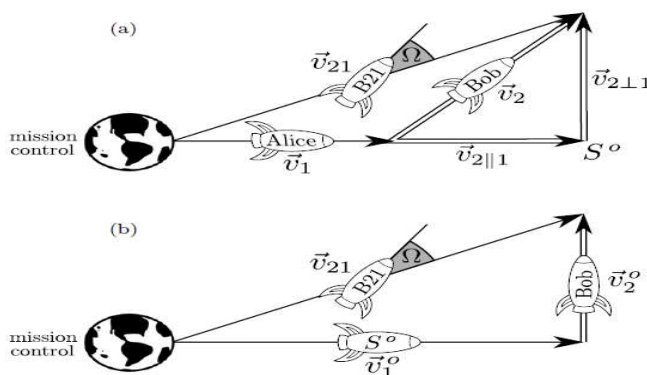


Figure 10: Parallel and Perpendicular Decompose the Velocities is Illustrated in Subfigures a) and b) Respectively

In subfigure 10a), we decompose the velocity \vec{v}_2 of Bob as measured by Alice into the components $\vec{v}_{2\parallel 1}$ and $\vec{v}_{2\perp 1}$, parallel and perpendicular to \vec{v}_1 respectively. The relativistic ally combined velocity \vec{v}_{21} is the velocity of Bob as seen by mission control. In subfigure 10b), we see the S^o frame, which is observed to have velocity \vec{v}_1^o by mission control, which represents the relativistic combination of velocities $\vec{v}_{2\parallel 1}$ and \vec{v}_1 . In the S^o frame Bob is measured to have velocity \vec{v}_2^o , perpendicular to \vec{v}_1^o .

As $\vec{v}_{2\parallel 1}$ and \vec{v}_1 are collinear by (10) the velocity of S^o as measured by mission control is

$$\vec{v}_1^o = \frac{\vec{v}_{2\parallel 1} + \vec{v}_1}{1 + \vec{v}_{2\parallel 1} \times \vec{v}_1} = \frac{\vec{v}_{2\parallel 1} + \vec{v}_1}{1 + \vec{v}_1 \cdot \vec{v}_2} \quad (12)$$

There is no Wigner rotation of the S^o frame relative to mission control. Therefore, we can think of a new situation, as illustrated in Figure 10b), where we have S^o moving at velocity \vec{v}_1^o relative to mission control, and Bob moving at some velocity \vec{v}_2^o as measured in the S^o frame. Since \vec{v}_1^o and \vec{v}_2^o are perpendicular then we can write,

$$\vec{v}_2^o = \gamma_{2\perp 1} \vec{v}_{2\perp 1} \quad (13)$$

$$\text{Where } \gamma_{2\parallel 1} = \frac{1}{\sqrt{1 - v_{2\parallel 1}^2}}$$

Thus, as mission control sees S^o moving at velocity \vec{v}_1^o , and the observer S^o sees Bob to be moving at the perpendicular velocity $\vec{v}_2^o = \gamma_{2\perp 1} \vec{v}_{2\perp 1}$, we see that the velocity of Bob with respect to mission control is given by

$$\vec{v}_{21} = \vec{v}_1^o + \frac{\gamma_{2\perp 1}}{\gamma_1^o} \vec{v}_{2\perp 1} \quad (14)$$

Equation (12) can be written as,

$$\vec{\gamma}_1^o = \frac{1}{\sqrt{1 - (v_1^o)^2}} = \gamma_{2\parallel 1} \gamma_1 (1 + \vec{v}_1 \cdot \vec{v}_2) \quad (15)$$

And hence (14) becomes

$$\vec{v}_{21} = \frac{\vec{v}_1 + \vec{v}_{2\parallel 1} + \sqrt{1 - v_1^2} \vec{v}_{2\perp 1}}{1 + \vec{v}_1 \cdot \vec{v}_2}$$

$$\vec{v}_{21} = \frac{\gamma_1 \cdot \vec{v}_1 + \vec{v}_2 + (\gamma_1 - 1)(\vec{v}_1 \cdot \vec{v}_2) \frac{\vec{v}_1}{v_1^2}}{\gamma_1 (1 + \vec{v}_1 \cdot \vec{v}_2)}$$

$$\vec{v}_{21} = \frac{\vec{v}_1 + (1 - \gamma_1^{-1})\vec{v}_{2\parallel 1} + \gamma_1^{-1}\vec{v}_2}{1 + \vec{v}_1 \cdot \vec{v}_2} \quad (16)$$

Similarly, we find

$$\begin{aligned} \vec{v}_{12} &= \frac{\vec{v}_2 + \vec{v}_{1\parallel 2} + \sqrt{1 - v_2^2} \vec{v}_{1\perp 2}}{1 + \vec{v}_1 \cdot \vec{v}_2} \\ \vec{v}_{12} &= \frac{\gamma_2 \cdot \vec{v}_2 + \vec{v}_1 + (\gamma_2 - 1)(\vec{v}_1 \cdot \vec{v}_2) \frac{\vec{v}_2}{v_2^2}}{\gamma_2(1 + \vec{v}_1 \cdot \vec{v}_2)} \\ \vec{v}_{12} &= \frac{\vec{v}_2 + (1 - \gamma_2^{-1})\vec{v}_{1\perp 2} + \gamma_2^{-1}\vec{v}_1}{1 + \vec{v}_1 \cdot \vec{v}_2} \end{aligned} \quad (17)$$

similar procedure to

These are the most elementary formulae for the composition of general velocities. We use a

$$\text{consider the Wigner rotation, where we must have } |\vec{v}_{21}| = |\vec{v}_{12}| \cdot |\vec{v}_{21}| = |\vec{v}_{12}| = \frac{\sqrt{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2}}{1 + \vec{v}_1 \cdot \vec{v}_2} \quad (18)$$

As previously described for the perpendicular case, the Wigner rotation angle Ω is exactly the angle between \vec{v}_{21} and \vec{v}_{12} as measured by mission control.

The Wigner rotation Ω angle then follows from the cross-product of the vectors \vec{v}_{21} and \vec{v}_{12} . Using (16), (17) and (18) we can write,

$$\vec{v}_{21} \times \vec{v}_{12} = |\vec{v}_{21}| |\vec{v}_{12}| \sin \Omega \quad (19)$$

$$\begin{aligned} \frac{\vec{v}_1 + (1 - \gamma_1^{-1})\vec{v}_{2\parallel 1} + \gamma_1^{-1}\vec{v}_2}{1 + \vec{v}_1 \cdot \vec{v}_2} \times \frac{\vec{v}_2 + (1 - \gamma_2^{-1})\vec{v}_{1\perp 2} + \gamma_2^{-1}\vec{v}_1}{1 + \vec{v}_1 \cdot \vec{v}_2} &= \frac{\sqrt{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2}}{1 + \vec{v}_1 \cdot \vec{v}_2} \frac{\sqrt{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2}}{1 + \vec{v}_1 \cdot \vec{v}_2} \sin \Omega \\ \sin \Omega &= \frac{|(\vec{v}_2 + (1 - \gamma_2^{-1})\vec{v}_{1\perp 2} + \gamma_2^{-1}\vec{v}_1) \times (\vec{v}_1 + (1 - \gamma_1^{-1})\vec{v}_{2\parallel 1} + \gamma_1^{-1}\vec{v}_2)|}{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2} \\ \sin \Omega &= \frac{|(\vec{v}_2 + (1 - \gamma_2^{-1})\vec{v}_{1\perp 2} + \gamma_2^{-1}\vec{v}_1) \times (\vec{v}_1 + (1 - \gamma_1^{-1})\vec{v}_{2\parallel 1} + \gamma_1^{-1}\vec{v}_2)|}{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2} \end{aligned} \quad (20)$$

This can be simplified to

$$\sin \Omega = v_1 v_2 \sin \theta \frac{\left[(1 - \gamma_1^{-1} \gamma_2^{-1} + (\vec{v}_1 \cdot \vec{v}_2) \left(\frac{1}{1 - \gamma_1^{-1}} + \frac{1}{1 - \gamma_2^{-1}} \right) + \frac{(\vec{v}_1 \cdot \vec{v}_2)}{(1 - \gamma_1^{-1})(1 - \gamma_2^{-1})}) \right]}{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2} \quad (21)$$

Where θ is the angle between \vec{v}_1 and \vec{v}_2 as measured by Alice. However

$$\bar{\gamma}_{12} = \frac{1}{\sqrt{1 - \vec{v}_{12}^2}} = \gamma_1 \gamma_2 (1 + \vec{v}_1 \cdot \vec{v}_2)$$

rearranged to give

$$\cos \theta = \frac{\gamma_{12} - \gamma_1 \gamma_2}{v_1 v_2 \gamma_1 \gamma_2}$$

Hence (21) can be written as,

$$\sin \Omega = v_1 v_2 \frac{\left[(1 - \gamma_1^{-1} \gamma_2^{-1} + (\vec{v}_1 \cdot \vec{v}_2) \left(\frac{1}{1 - \gamma_1^{-1}} + \frac{1}{1 - \gamma_2^{-1}} \right) + \frac{(\vec{v}_1 \cdot \vec{v}_2)}{(1 - \gamma_1^{-1})(1 - \gamma_2^{-1})}) \right]}{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2} \sin \theta$$

$$\sin \Omega = \frac{v_1 v_2 \gamma_1 \gamma_2 (1 + \gamma_1 + \gamma_2 + \gamma_{12})}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} \sin \theta$$

$$\Omega = \sin^{-1} \left[\frac{v_1 v_2 \gamma_1 \gamma_2 (1 + \gamma_1 + \gamma_2 + \gamma_{12})}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} \sin \theta \right]$$

Which, some may recognize as Stapp's elegant formula [28]. Similarly, using the definition of the dot-product to find the Wigner rotation angle Ω , one finds

$$\cos \Omega = \frac{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2 \left[\left(\frac{1}{1 - \gamma_1^{-1}} + \frac{1}{1 - \gamma_2^{-1}} \right) - \frac{(\vec{v}_1 \cdot \vec{v}_2)}{(1 - \gamma_1^{-1})(1 - \gamma_2^{-1})} \right]}{|\vec{v}_1 + \vec{v}_2|^2 - |\vec{v}_1 \times \vec{v}_2|^2}$$

$$\cos \Omega + 1 = \frac{(\gamma_{12} + \gamma_1 + \gamma_2 + 1)^2}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)}$$

$$\cos \Omega = \frac{(\gamma_{12} + \gamma_1 + \gamma_2 + 1)^2}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} - 1$$

$$\Omega = \cos^{-1} \left[\frac{(\gamma_{12} + \gamma_1 + \gamma_2 + 1)^2}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} - 1 \right] \quad (22)$$

Equation (22) is the Wigner rotation angle Ω between the Bob and Alice.

4.2 Numerical Calculation

Let us consider the Alice velocity $0.4C$, which is constant, and the different velocity of Bob and corresponding Wigner rotation values calculated using equation (22) in the following table.

Table 1

Bob Velocity	Wigner Rotation
0.3c	4.32^0
0.4c	4.83^0
0.5c	6.40^0
0.6c	7.98^0
0.7c	9.84^0
0.8c	11.89^0
0.9c	14.89^0

The Alice velocity is constant $0.4C$ and the change of Bob velocity with respect to change of Wigner rotation as shown in the figure 11.

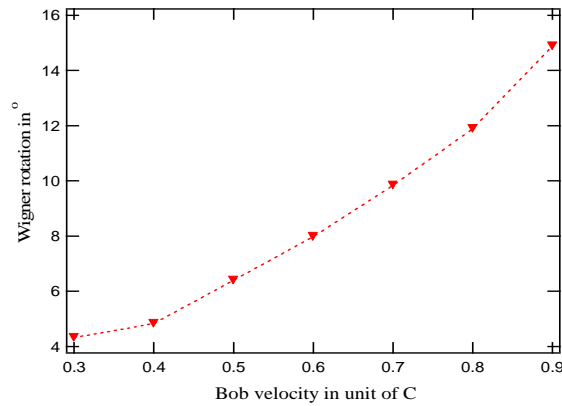


Figure 11: The Change of Wigner Rotation with Respect to Different Bob Velocity

4.3 Method Two Calculation of Wigner Rotation

The Wigner rotation can also be calculated through the following process. In figure 1, \vec{W}' be the velocity of muon with respect to lab frame then according to the velocity addition formula for the most general Lorentz transformation [8-10] we can write

$$\vec{W}'' = \vec{U} \oplus \vec{V} = \frac{\vec{V} + \vec{U} \left[\frac{(\vec{U} \cdot \vec{V})}{U^2} (\gamma_U - 1) + \gamma_U \right]}{\gamma_U (1 + \vec{U} \cdot \vec{V})} \quad (23)$$

Now, if muon moves with velocity \vec{W}'' with respect to lab frame and electron moves with velocity \vec{W} respect to muon, then according to the velocity addition, formula for the most general Lorentz transformation [13] we can write

$$\begin{aligned}\vec{W}'' = \vec{W}'' \oplus \vec{W} &= \frac{\vec{W} + \vec{W}'' \left\{ \frac{(\vec{W}'' \cdot \vec{W})}{W''^2} (\gamma_{\vec{W}''} - 1) + \gamma_{\vec{W}''} \right\}}{\gamma_{\vec{W}''} (1 + \vec{W}'' \cdot \vec{W})} \\ \text{or, } (\vec{U} \oplus \vec{V}) \oplus \vec{W} &= \frac{\vec{W} + (\vec{U} \oplus \vec{V}) \left\{ \frac{(\vec{U} \oplus \vec{V}) \cdot \vec{W}}{(\vec{U} \oplus \vec{V})^2} (\gamma_{(\vec{U} \oplus \vec{V})} - 1) + \gamma_{(\vec{U} \oplus \vec{V})} \right\}}{\gamma_{(\vec{U} \oplus \vec{V})} (1 + (\vec{U} \oplus \vec{V}) \cdot \vec{W})}\end{aligned}\quad (24)$$

Again, from muon and electron according to the velocity addition formula for most general Lorentz transformation [29-30] the resultant velocity of \vec{V} and \vec{W} can be written as

$$\vec{W}' = \vec{V} \oplus \vec{W} = \frac{\vec{W} + \vec{V} \left\{ \frac{(\vec{V} \cdot \vec{W})}{V^2} (\gamma_V - 1) - \gamma_V \right\}}{\gamma_V (1 + \vec{V} \cdot \vec{W})}\quad (25)$$

Finally, if electron moves with velocity \vec{W}' with respect to pion, then the resultant velocity [29] electron with respect to lab frame can be written as

$$\begin{aligned}\vec{W}'' = \vec{U} \oplus \vec{W}' &= \frac{\vec{W}' + \vec{U} \left\{ \frac{(\vec{U} \cdot \vec{W}')}{U^2} (\gamma_U - 1) - \gamma_U \right\}}{\gamma_U (1 + \vec{U} \cdot \vec{W}')} \\ \text{or, } \vec{U} \oplus (\vec{V} \oplus \vec{W}) &= \frac{(\vec{V} \oplus \vec{W}) + \vec{U} \left\{ \frac{\vec{U} \cdot (\vec{V} \oplus \vec{W})}{U^2} (\gamma_U - 1) - \gamma_U \right\}}{\gamma_U (1 + (\vec{V} \oplus \vec{W}) \cdot \vec{U})}\end{aligned}\quad (26)$$

Specifically, to illustrate Wigner rotation for velocity vectors in unit of c are defined as,

$\vec{U} = (u_x, u_y, 0) = (0.5, 0.2, 0.0)$ Velocity of pion relative to lab frame;

$\vec{V} = (v_x, v_y, 0) = (0.3, 0.5, 0.0)$ Velocity of muon relative to pion;

$\vec{W} = (w_x, w_y, 0) = (0.4, 0.2, 0.0)$ Velocity of electron relative to muon.

The corresponding γ factors are as follows

$$\gamma_u = \frac{1}{\sqrt{1-U^2}} = 1.18678, \quad \gamma_v = \frac{1}{\sqrt{1-V^2}} = 1.23091, \quad \gamma_{U \oplus V} = \frac{1}{\sqrt{1-(\vec{U} \oplus \vec{V})^2}} = 8.45154$$

From equation (23) and (24) we have $(\vec{U} \oplus \vec{V}) \oplus \vec{W} = \begin{bmatrix} 0.7237 \\ 0.96175 \\ 0 \end{bmatrix} = \vec{A} \text{ (say)}$

From equation (25) and (26) we get $\vec{U} \oplus (\vec{V} \oplus \vec{W}) = \begin{bmatrix} 6.65228 \\ 5.17574 \\ 0 \end{bmatrix} = \vec{B}$ (say)

$$\vec{A} \cdot \vec{B} = AB \cos \Omega$$

We know that $\Omega = \cos^{-1} \left(\frac{\vec{A} \cdot \vec{B}}{AB} \right)$

Where $\vec{A} = 0.7237i + 0.96175j + 0k$, $A = \sqrt{(0.7237)^2 + (0.96175)^2} = 1.20362$

And $\vec{B} = 6.65228i + 5.17574j + 0k$, $B = \sqrt{(6.65228)^2 + (5.17574)^2} = 8.428589$

$$\vec{A} \cdot \vec{B} = 9.792$$

Hence, $\Omega = \cos^{-1}(0.96) = 16.26^\circ$

Again let,

$$\vec{U} = (u_x, u_y, 0) = (0.5, 0.2, 0.0), \vec{V} = (v_x, v_y, 0) = (0.2, 0.4, 0.0), \vec{W} = (w_x, w_y, 0) = (0.5, 0.3, 0.0) \text{ \& } \\ \vec{U} = (u_x, u_y, 0) = (0.4, 0.3, 0.0), \vec{V} = (v_x, v_y, 0) = (0.2, 0.5, 0.0), \vec{W} = (w_x, w_y, 0) = (0.6, 0.3, 0.0)$$

Be two sets of velocity vectors of pion decay chain $\pi \rightarrow \mu \rightarrow e$, as figure 1 then using equations (23),(24),(25) and (26) we have the velocity vectors of electron relative to lab frame, $(\vec{U} \oplus \vec{V}) \oplus \vec{W}$ and $\vec{U} \oplus (\vec{V} \oplus \vec{W})$ are

$$\begin{bmatrix} 1.00669 \\ 0.7605 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.9978 \\ 0.575 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0.9708 \\ 0.9487 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.74188 \\ 0.680924 \\ 0 \end{bmatrix} \text{ respectively.}$$

Using similar process as previous we have the Wigner rotations in this case

$$\Omega = \cos^{-1}(0.99296) = 7.17^\circ \text{ (App.) and } \Omega = \cos^{-1}(0.9995) = 1.8^\circ \text{ (App.) respectively.}$$

5. CONCLUSIONS

We have derived the formula of Wigner rotation in two different ways. We have observed that there is no Wigner rotation for two linear Lorentz boosts. We have calculated the numerical values of Wigner rotation for different cases. We have observed that the Wigner rotation increases due to the increase of velocity.

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